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A spherical harmonic expansion of the Boltzmann equation for nonhydrodynamic weakly ionized plasma in the presence of both electric and magnetic fields

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Abstract

The Boltzmann kinetic equation for nonhydrodynamic weakly ionized plasma in the presence of both electric and magnetic fields is considered. The charged particles distribution function is decomposed in terms of spherical harmonics in momentum space. After substituting the expansion into the Boltzmann equation an infinite hierarchy of differential equations for the distribution function expansion coefficients is derived. The cases of Cartesian, cylindrical and spherical coordinates in configuration space are studied. We applied obtained equations to the description of electron transport in nitrogen at high values of rf electric field intensity to number density ratio E/N.

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1. Introduction

The relativistic Boltzmann kinetic equation for the phase-space distribution function $f(\boldsymbol{r}, \boldsymbol{p}, t)$ of charged particles (electrons or ions) is expressed as

$$(\partial_t + \boldsymbol{v} \cdot \partial_{\boldsymbol{r}} + \boldsymbol{F} \cdot \partial_{\boldsymbol{p}}) f(\boldsymbol{r}, \boldsymbol{p}, t) = C(f), \tag{1}$$

where $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ is the Lorentz force acting on charge q; $\boldsymbol{E}$ and $\boldsymbol{B}$ are electric and magnetic fields, respectively; $C(f)$ is the collision operator that acts in momentum space and characterizes the change rate of the distribution function due to collisions of charged particles with neutral atoms and molecules. The momentum $\boldsymbol{p}$ and velocity $\boldsymbol{v}$ are connected by the relation $\boldsymbol{p} = M\boldsymbol{v}[1 - (v/c)^2]^{-1/2}$, where c is the light velocity and M is the particle mass.

The relativistic Boltzmann equation for electrons in air was solved numerically in articles [1–3]. In article [1], the Boltzmann equation for a population of relativistic electrons in a constant external electric field in air was solved numerically in order to determine parameters of the runway electron beam. In that paper a significant decrease in the air breakdown threshold under conditions in which a small number of relativistic electrons were present in the air was demonstrated. In paper [2], a new mechanism of breakdown in air—runaway breakdown—was applied to explain the observation of multiple lightning discharges observed in the Ivy-Mike thermonuclear test in the early 1950s. In paper [3], a three-dimensional Monte Carlo simulation of runaway relativistic electron breakdown in air in the presence of static electric and magnetic fields with application to red sprites and terrestrial gamma ray flashes was given.

Consider spherical coordinates in momentum space $\boldsymbol{p} = (p, \theta_p, \varphi_p)$. The decomposition of the distribution function in terms of spherical harmonics in momentum space is described by

$$f(\boldsymbol{r};\, p, \theta_p, \varphi_p;\, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=0}^{1} f_{n,m,s}(\boldsymbol{r};\, p;\, t) Y_{n,m,s}(\theta_p, \varphi_p), \tag{2}$$

where [4, 5]

$$Y_{n,m,s}(\theta_p, \varphi_p) = P_n^m(\cos\theta_p)[\delta_{s,1}\sin(m\varphi_p) + \delta_{s,0}\cos(m\varphi_p)], \tag{3}$$

with the orthogonality relationship

$$\int_0^{\pi}\int_0^{2\pi} Y_{n,m,s}(\theta_p, \varphi_p) Y_{n',m',s'}(\theta_p, \varphi_p) \sin\theta_p \,\mathrm{d}\theta_p \,\mathrm{d}\varphi_p = \frac{2\pi}{1+2n} \frac{(n+m)!}{(n-m)!}(1+\delta_{m,0})\delta_{n,n'}\delta_{m,m'}\delta_{s,s'}, \tag{4}$$

$\delta_{i,j}$ is the Kronecker symbol. The associated Legendre functions are defined by

$$P_n^m(\mu) = \frac{1}{2^n n!}(1-\mu^2)^{m/2} \frac{\mathrm{d}^{n+m}}{\mathrm{d}\mu^{n+m}}(\mu^2-1)^n. \tag{5}$$

In weakly ionized plasma elastic and inelastic collisions of charged particles with only neutral atoms and molecules are taken into account. In this case the decomposition of the collision integral in terms of spherical harmonics is written as [6, 7]

$$C(f) = \sum_{s=0}^{1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} C_n(f_{n,m,s}) Y_{n,m,s}(\theta_p, \varphi_p). \tag{6}$$

Our study does not assume any specific form of the collisional operator, and our results can be applied for different cases of followed processes and different model expressions for them. The principal objective was to derive an infinite hierarchy of equations for the coefficients $f_{n,m,s}$ which can be used to numerically solve the Boltzmann equation with any approximation order. Specific expressions for $C_n(f_{n,m,s})$ can be obtained from cross sections of physical processes taken into account by procedures described in the literature [4]. The definition of coefficients $C_n(f_{n,m,s})$ from cross sections of charged particle collisions with neutral atoms and molecules was considered by several authors [4, 7–9]. For example, in the article by Porokhova *et al* [7] coefficients $C_n(f_{n,m,s})$ were defined for electron collisions with atoms of an inert gas. Isotropic cross sections for inelastic scattering and anisotropic cross sections for elastic scattering were used. In papers [8, 9], the decomposition of the collision integral in terms of Legendre polynomials was made. Coefficients $C_n(f_n)$ were defined for electron collisions with nitrogen molecules. Anisotropic cross sections were used for elastic and inelastic collisions. Electron–electron collisions were taken into account in collisional integrals in papers [4, 10, 11]. The collision integral for relativistic electrons in neutral gases was derived by Gurevich *et al* [12].

Jonston [5] supposed that it was impossible to obtain the infinite hierarchy of equations for coefficients $f_{n,m,s}$ by substituting the decomposition (2) into the Boltzmann equation (1) and using the orthogonality relationship (4). The arisen difficulties that hindered this problem are due to terms such as $\mu(1-\mu)^{1/2}\partial_\mu P_n^m(\mu)$ for which no simple recursion relation was obtained. To eliminate this difficulty Jonston proposed to use a tensor scalar product expansion of the distribution function that is equivalent to the distribution function decomposition in spherical harmonics. By using this formalism a system of 16 differential equations for the distribution function expansion coefficients for terms up to $n = 3$ was obtained for the Cartesian configuration space in [5]. As pointed out in [7] it seems impossible to get more than four-term approximation using this formalism.

The Boltzmann equation in the presence of an electric field was reformulated entirely in terms of spherical tensors by Robson and Ness [13] for the Cartesian geometry and hydrodynamic plasma conditions. They derived the hierarchy of equations for tensor decomposition coefficients. The extension of this formalism to the case of electric and magnetic fields was obtained by Ness [14]. Special configurations of the magnetic field parallel and perpendicular to the electric field were discussed in detail.

Robson *et al* [6] generalized the results obtained in [13, 14] to nonhydrodynamic plasma conditions. The infinite hierarchy of equations for the distribution function expansion coefficients in the presence of electric field $\boldsymbol{E} = (E_r, 0, E_z)$ and gradient $\partial_{\boldsymbol{r}} = (\partial_r, 0, \partial_z)$ was presented for the axially symmetric cylindrical geometry in [6]. The case of perpendicular electric $\boldsymbol{E} = (E_r, 0, 0)$ and magnetic $\boldsymbol{B} = (0, 0, B_z)$ fields and gradient $\partial_{\boldsymbol{r}} = (\partial_r, 0, 0)$ was derived in [7].

The infinite hierarchy of equations for expansion coefficients in the presence of radial electric field $\boldsymbol{E} = (E_r, 0, 0)$ and gradient $\partial_{\boldsymbol{r}} = (\partial_r, 0, 0)$ was also obtained in [6] for spherically symmetric geometry. The same for the electric field $\boldsymbol{E} = (0, 0, E_z)$ and gradient $\partial_{\boldsymbol{r}} = (0, 0, \partial_z)$ was derived for Cartesian geometry.

The purpose of this paper is to demonstrate that the infinite hierarchy of equations for distribution function expansion coefficients can be obtained in a quite simple way with no reference to the formalism given in [5, 13, 14] by only using properties of associated Legendre functions. The derived infinite hierarchy of differential equations for distribution function expansion coefficients can be truncated at any desired step. Such a finite system of equations allows us to numerically solve the Boltzmann equation with any approximation order. We would like to underline that terms such as $\mu(1-\mu)^{1/2}\partial_\mu P_n^m(\mu)$ can be transformed into a linear combination of associated Legendre functions, which was claimed impossible in Jonston's paper where he had to choose another way of solving the problem [see [5], p 1106].

In section 2, the infinite hierarchy of equations for distribution function expansion coefficients in the presence of electric $\boldsymbol{E} = (E_x, E_y, E_z)$ and magnetic $\boldsymbol{B} = (B_x, B_y, B_z)$ fields, and gradients $\partial_{\boldsymbol{r}} = (\partial_x, \partial_y, \partial_z)$ is deduced for Cartesian coordinates $\boldsymbol{r} = (x, y, z)$.

Similar expressions for cylindrical and spherical coordinates are given in sections 3 and 4. Comparison of our results with results obtained by other researchers is given in section 5. We apply obtained equations to describe electron transport in nitrogen at high values of E/N in section 6.

2. The hierarchy of equations for Cartesian geometry

Let $\boldsymbol{r} = (x, y, z)$ and $\boldsymbol{p} = (p, \theta_p, \varphi_p)$ denote Cartesian coordinates in configuration space and spherical coordinates in momentum space, respectively. All components of the electric and magnetic fields $\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{B} = (B_x, B_y, B_z)$ are given. Unit vectors $\boldsymbol{i}_x$ and $\boldsymbol{i}_z$ define directions of axes x and z, respectively. The unit vector $\boldsymbol{i}_p$ defines the direction of the momentum $\boldsymbol{p}$. In this frame $\cos\theta_p = \boldsymbol{i}_z \cdot \boldsymbol{i}_p$. The angle φ_p is the dihedral angle between planes formed by the vectors $\boldsymbol{i}_z$ and $\boldsymbol{i}_x$ and the vectors $\boldsymbol{i}_z$ and $\boldsymbol{i}_p$. In this phase space the Boltzmann equation (1) is written as

$$\left(\partial_t + v_x\partial_x + v_y\partial_y + v_z\partial_z + F_p\partial_p + F_{\theta_p}\frac{1}{p}\partial_{\theta_p} + F_{\varphi_p}\frac{1}{p\sin\theta_p}\partial_{\varphi_p}\right) \times f(x, y, z; p, \theta_p, \varphi_p, t) = C(f). \tag{7}$$

The Cartesian components of the Lorentz force acting on charge q are equal to

$$\begin{aligned} F_x &= q(v_y B_z - v_z B_y + E_x),\\ F_y &= q(v_z B_x - v_x B_z + E_y),\\ F_z &= q(v_x B_y - v_y B_x + E_z). \end{aligned} \tag{8}$$

The spherical components of the velocity vector are related to the Cartesian ones by

$$v_x = v\sin\theta_p\cos\varphi_p, \qquad v_y = v\sin\theta_p\sin\varphi_p, \qquad v_z = v\cos\theta_p. \tag{9}$$

The Cartesian components of the Lorentz force vector are related to the spherical ones by

$$\begin{aligned} F_p &= (F_x\cos\varphi_p + F_y\sin\varphi_p)\sin\theta_p + F_z\cos\theta_p,\\ F_{\theta_p} &= (F_x\cos\varphi_p + F_y\sin\varphi_p)\cos\theta_p - F_z\sin\theta_p,\\ F_{\varphi_p} &= -F_x\sin\varphi_p + F_y\cos\varphi_p. \end{aligned} \tag{10}$$

The substitution of the distribution function expansion (2), the collision integral expansion (6) and expressions (8)–(10) into the Boltzmann equation (7) results in

$$\begin{aligned} &\{\partial_t + v(1-\mu^2)^{1/2}[\cos\varphi_p\partial_x + \sin\varphi_p\partial_y] + v\mu\partial_z\\ &\quad + [qE_x(1-\mu^2)^{1/2}\cos\varphi_p + qE_y(1-\mu^2)^{1/2}\sin\varphi_p + qE_z\mu]\partial_p\\ &\quad - [\omega_x\sin\varphi_p - \omega_y\cos\varphi_p + qE_x p^{-1}\mu\cos\varphi_p + qE_y p^{-1}\mu\sin\varphi_p\\ &\quad - qE_z p^{-1}(1-\mu^2)^{1/2}](1-\mu^2)^{1/2}\partial_\mu + [\omega_x\mu\cos\varphi_p + \omega_y\mu\sin\varphi_p\\ &\quad - (1-\mu^2)^{1/2}\omega_z - qE_x p^{-1}\sin\varphi_p + qE_y p^{-1}\cos\varphi_p](1-\mu^2)^{-1/2}\partial_{\varphi_p}\}\\ &\quad \times \sum_{s=0}^{1}\sum_{n=0}^{\infty}\sum_{m=0}^{n} f_{n,m,s}P_n^m(\mu)[\delta_{s,1}\sin(m\varphi_p) + \delta_{s,0}\cos(m\varphi_p)]\\ &= \sum_{s=0}^{1}\sum_{n=0}^{\infty}\sum_{m=0}^{n} C_n(f_{n,m,s})Y_{n,m,s}, \end{aligned} \tag{11}$$

where

$$\mu = \cos\theta_p, \qquad \omega_x = \frac{qv}{p}B_x, \qquad \omega_y = \frac{qv}{p}B_y, \qquad \omega_z = \frac{qv}{p}B_z.$$

Replace expressions

$$\begin{aligned} &(1-\mu^2)^{1/2}P_n^m, \quad (1-\mu^2)^{-1/2}P_n^m, \quad \mu P_n^m, \quad (1-\mu^2)^{1/2}\partial_\mu P_n^m,\\ &\mu(1-\mu^2)^{1/2}\partial_\mu P_n^m, \quad (1-\mu^2)\partial_\mu P_n^m, \quad \mu(1-\mu^2)^{-1/2}P_n^m, \end{aligned} \tag{12}$$

by linear combinations of associated Legendre functions according to formulae (A.7)–(A.11), (A.16)–(A.17) and substitute results into equation (11).

After multiplication of equation (11) by $Y_{n',m',s'}(\theta_p,\varphi_p)$ and integration over all directions of momentum space $\boldsymbol{i}_p$, using the orthogonality relationship (4), the following infinite hierarchy of equations for the distribution function coefficients $f_{n,m,s}$ is obtained:

$$\begin{aligned}
\partial_t f_{n,m,s} &+ \frac{1+\delta_{m,1}}{2(2n-1)}\left[\left(v\partial_x + qE_x\partial^{(-)}_{p,n}\right) f_{n-1,m-1,s} - (-1)^s\left(v\partial_y + qE_y\partial^{(-)}_{p,n}\right) f_{n-1,m-1,1-s}\right]\\
&- \frac{(n-m-1)(n-m)}{2(2n-1)}\left[\left(v\partial_x + qE_x\partial^{(-)}_{p,n}\right) f_{n-1,m+1,s}\right.\\
&\left.+ (-1)^s\left(v\partial_y + qE_y\partial^{(-)}_{p,n}\right) f_{n-1,m+1,1-s}\right] + \frac{n-m}{2n-1}\left[v\partial_z + qE_z\partial^{(-)}_{p,n}\right] f_{n-1,m,s}\\
&- \frac{1+\delta_{m,1}}{2(2n+3)}\left[\left(v\partial_x + qE_x\partial^{(+)}_{p,n}\right) f_{n+1,m-1,s} - (-1)^s\left(v\partial_y + qE_y\partial^{(+)}_{p,n}\right) f_{n+1,m-1,1-s}\right]\\
&+ \frac{(n+m+1)(n+m+2)}{2(2n+3)}\left[\left(v\partial_x + qE_x\partial^{(+)}_{p,n}\right) f_{n+1,m+1,s}\right.\\
&\left.+ (-1)^s\left(v\partial_y + qE_y\partial^{(+)}_{p,n}\right) f_{n+1,m+1,1-s}\right] + \frac{n+m+1}{2n+3}\left[v\partial_z + qE_z\partial^{(+)}_{p,n}\right] f_{n+1,m,s}\\
&+ \frac{1+\delta_{m,1}}{2}[\omega_y f_{n,m-1,s} + (-1)^s\omega_x f_{n,m-1,1-s}] - (-1)^s m\omega_z f_{n,m,1-s}\\
&- \frac{(n-m)(n+m+1)}{2}[\omega_y f_{n,m+1,s} - (-1)^s\omega_x f_{n,m+1,1-s}] = C_n(f_{n,m,s}),\\
&(n = 0,1,2,\ldots,\infty;\quad m = 0,1,2,\ldots,n;\quad s = 0,1).
\end{aligned}\tag{13}$$

The following designations are used

$$\partial^{(-)}_{p,n} = \partial_p - \frac{n-1}{p},\qquad \partial^{(+)}_{p,n} = \partial_p + \frac{n+2}{p}.$$

In equation (13) we assume (see expression (A.6)) that

$$f_{n,m,s} = 0,\quad \text{for cases:}\quad n<0\quad\text{or}\quad m<0\quad\text{or}\quad n<m\quad\text{or}\quad m=0, s=1.\tag{14}$$

Truncation of the expansion (2) and the hierarchy (13) at $n = n_{\max}$ corresponds to the $(n_{\max}+1)$-term approximation and results in $(n_{\max}+1)^2$ independent equations and the same number of unknown $f_{n,m,s}$.

3. The hierarchy of equations for cylindrical geometry

Let $\boldsymbol{r} = (r,\varphi,z)$ and $\boldsymbol{p} = (p,\theta_p,\varphi_p)$ denote cylindrical coordinates in configuration space and moving spherical coordinates in momentum space, respectively. The unit vector $\boldsymbol{i}_z$ defines the axis z-direction. Unit vectors $\boldsymbol{i}_p$ and $\boldsymbol{i}_r$ define directions of momentum $\boldsymbol{p}$ and radius vector $\boldsymbol{r}$, respectively. In this frame $\cos\theta_p = \boldsymbol{i}_z\cdot\boldsymbol{i}_p$. The angle φ_p is the dihedral angle between planes formed by vectors $\boldsymbol{i}_z$ and $\boldsymbol{i}_r$ and vectors $\boldsymbol{i}_z$ and $\boldsymbol{i}_p$. This coordinate system is commonly used for neutron transport problems [15]. All components of electric $\boldsymbol{E} = (E_r, E_\varphi, E_z)$ and magnetic $\boldsymbol{B} = (B_r, B_\varphi, B_z)$ fields are given. The Boltzmann equation (1) in the chosen coordinates is

$$\begin{aligned}
&\left[\partial_t + v_r\partial_r + \frac{v_\varphi}{r}\left(\partial_\varphi - \partial_{\varphi_p}\right) + v_z\partial_z + F_p\partial_p + F_{\theta_p}\frac{1}{p}\partial_{\theta_p} + \frac{F_{\varphi_p}}{p\sin\theta_p}\partial_{\varphi_p}\right]\\
&\qquad\times f(r,\varphi,z;p,\theta_p,\varphi_p,t) = C(f).
\end{aligned}\tag{15}$$

The cylindrical components of the Lorentz force acting on charge q are equal to

$$F_r = q(v_\varphi B_z - v_z B_\varphi + E_r),$$
$$F_\varphi = q(v_z B_r - v_r B_z + E_\varphi), \qquad (16)$$
$$F_z = q(v_r B_\varphi - v_\varphi B_r + E_z).$$

The spherical components of the velocity vector are related to the cylindrical ones by

$$v_r = v\sin\theta_p\cos\varphi_p, \qquad v_\varphi = v\sin\theta_p\sin\varphi_p, \qquad v_z = v\cos\theta_p. \qquad (17)$$

The cylindrical components of the Lorentz force vector are related to the spherical ones by

$$F_p = (F_r\cos\varphi_p + F_\varphi\sin\varphi_p)\sin\theta_p + F_z\cos\theta_p,$$
$$F_{\theta_p} = (F_r\cos\varphi_p + F_\varphi\sin\varphi_p)\cos\theta_p - F_z\sin\theta_p, \qquad (18)$$
$$F_{\varphi_p} = -F_r\sin\theta_p + F_\varphi\cos\varphi_p.$$

Substituting the distribution function expansion (2), the collision integral expansion (6) and expressions (16)–(18) into the Boltzmann equation (15) gives the following equation:

$$\begin{aligned}
&\{\partial_t + v(1-\mu^2)^{1/2}[\cos\varphi_p\partial_r + r^{-1}\sin\varphi_p(\partial_\varphi - \partial_{\varphi_p})] + v\mu\partial_z \\
&\quad + [(1-\mu^2)^{1/2}(qE_r\cos\varphi_p + qE_\varphi\sin\varphi_p) + \mu qE_z]\partial_p - [\omega_r\sin\varphi_p - \omega_\varphi\cos\varphi_p \\
&\quad + \mu p^{-1}(qE_r\cos\varphi_p + qE_\varphi\sin\varphi_p) - qE_z p^{-1}(1-\mu^2)^{1/2}](1-\mu^2)^{1/2}\partial_\mu \\
&\quad + [\mu(\omega_r\cos\varphi_p + \omega_\varphi\sin\varphi_p) - (1-\mu^2)^{1/2}\omega_z \\
&\quad - qE_r p^{-1}\sin\varphi_p + qE_\varphi p^{-1}\cos\varphi_p] \\
&\quad \times (1-\mu^2)^{-1/2}\partial_{\varphi_p}\}\sum_{s=0}^{1}\sum_{n=0}^{\infty}\sum_{m=0}^{n} f_{n,m,s}P_n^m(\mu)[\delta_{s,1}\sin(m\varphi_p) + \delta_{s,0}\cos(m\varphi_p)] \\
&= \sum_{s=0}^{1}\sum_{n=0}^{\infty}\sum_{m=0}^{n} C_n(f_{n,m,s})Y_{n,m,s}, \qquad (19)
\end{aligned}$$

where

$$\omega_r = qvp^{-1}B_r, \qquad \omega_\varphi = qvp^{-1}B_\varphi, \qquad \omega_z = qvp^{-1}B_z.$$

Replace expressions (12) by the linear combination of associated Legendre functions according to formulae (A.7)–(A.11), (A.16)–(A.17) and substitute results into equation (19). Multiplying equation (19) by $Y_{n',m',s'}(\theta_p, \varphi_p)$, integrating over all directions of momentum space $\boldsymbol{i}_p$ and using the orthogonality relationship (4) give the following infinite hierarchy of equations for coefficients $f_{n,m,s}$:

$$\begin{aligned}
&\partial_t f_{n,m,s} + \frac{1+\delta_{m,1}}{2(2n-1)}\left[\left(v\partial_r - (m-1)\frac{v}{r} + qE_r\partial_{p,n}^{(-)}\right)f_{n-1,m-1,s}\right. \\
&\quad \left. - (-1)^s\left(\frac{v}{r}\partial_\varphi + qE_\varphi\partial_{p,n}^{(-)}\right)f_{n-1,m-1,1-s}\right] \\
&\quad - \frac{(n-m-1)(n-m)}{2(2n-1)}\left[\left(v\partial_r + (m+1)\frac{v}{r} + qE_r\partial_{p,n}^{(-)}\right)f_{n-1,m+1,s}\right. \\
&\quad \left. + (-1)^s\left(\frac{v}{r}\partial_\varphi + qE_\varphi\partial_{p,n}^{(-)}\right)f_{n-1,m+1,1-s}\right] + \frac{n-m}{2n-1}\left[v\partial_z + qE_z\partial_{p,n}^{(-)}\right]f_{n-1,m,s} \\
&\quad - \frac{1+\delta_{m,1}}{2(2n+3)}\left[\left(v\partial_r - (m-1)\frac{v}{r} + qE_r\partial_{p,n}^{(+)}\right)f_{n+1,m-1,s}\right.
\end{aligned}$$

$$- (-1)^s \left(\frac{v}{r} \partial_\varphi + q E_\varphi \partial^{(+)}_{p,n} \right) f_{n+1,m-1,1-s} \bigg]$$
$$+ \frac{(n+m+1)(n+m+2)}{2(2n+3)} \left[\left(v\partial_r + (m+1)\frac{v}{r} + qE_r \partial^{(+)}_{p,n} \right) f_{n+1,m+1,s} \right.$$
$$\left. + (-1)^s \left(\frac{v}{r} \partial_\varphi + q E_\varphi \partial^{(+)}_{p,n} \right) f_{n+1,m+1,1-s} \right] + \frac{n+m+1}{2n+3} \left[v\partial_z + qE_z \partial^{(+)}_{p,n} \right] f_{n+1,m,s}$$
$$+ \frac{1+\delta_{m,1}}{2} [\omega_\varphi f_{n,m-1,s} + (-1)^s \omega_r f_{n,m-1,1-s}] - (-1)^s m\omega_z f_{n,m,1-s}$$
$$- \frac{(n-m)(n+m+1)}{2} [\omega_\varphi f_{n,m+1,s} - (-1)^s \omega_r f_{n,m+1,1-s}] = C_n(f_{n,m,s}),$$
$$(n = 0, 1, 2, \ldots, \infty; \quad m = 0, 1, 2, \ldots, n; \quad s = 0, 1). \tag{20}$$

Coefficients $f_{n,m,s}$ satisfy condition (14).

For the four-term approximation case, 16 equations (B.1)–(B.16) are given in appendix B.

4. The hierarchy of equations for spherical geometry

Let $\boldsymbol{r} = (r, \theta, \varphi)$ and $\boldsymbol{p} = (p, \theta_p, \varphi_p)$ denote spherical coordinates in configuration space and moving spherical coordinates in momentum space, respectively. The unit vector $\boldsymbol{i}_z$ defines the axis z direction. Unit vectors $\boldsymbol{i}_p$ and $\boldsymbol{i}_r$ define directions of momentum $\boldsymbol{p}$ and radius vector $\boldsymbol{r}$, respectively. In this frame $\cos\theta_p = \boldsymbol{i}_r \cdot \boldsymbol{i}_p$. The angle φ_p is the dihedral angle between the planes formed by vectors $\boldsymbol{i}_r$ and $\boldsymbol{i}_z$ and vectors $\boldsymbol{i}_r$ and $\boldsymbol{i}_p$. Such a coordinate system is commonly used for neutron transport problems [15]. All components of electric $\boldsymbol{E} = (E_r, E_\theta, E_\varphi)$ and magnetic $\boldsymbol{B} = (B_r, B_\theta, B_\varphi)$ fields are given. The Boltzmann equation (1) in the chosen phase space has the following form:

$$\left\{ \partial_t + v_r \partial_r + \frac{v_\theta}{r} \partial_\theta + \frac{v_\varphi}{r \sin\theta} \left[\partial_\varphi - \cos\theta \, \partial_{\varphi_p} \right] - \frac{1}{r} \left(v_\theta^2 + v_\varphi^2 \right)^{1/2} \partial_{\theta_p} \right.$$
$$\left. + F_p \partial_p + \frac{F_{\theta_p}}{p} \partial_{\theta_p} + \frac{F_{\varphi_p}}{p \sin\theta_p} \partial_{\varphi_p} \right\} f(r, \theta, \varphi; p, \theta_p, \varphi_p; t) = C(f). \tag{21}$$

The spherical components of the Lorentz force acting on charge q are described by

$$\begin{aligned} F_r &= q(v_\theta B_\varphi - v_\varphi B_\theta + E_r), \\ F_\theta &= q(v_\varphi B_r - v_r B_\varphi + E_\theta), \\ F_\varphi &= q(v_r B_\theta - v_\theta B_r + E_\varphi). \end{aligned} \tag{22}$$

The spherical components of the velocity vector are

$$v_r = v \cos\theta_p, \qquad v_\theta = v \sin\theta_p \cos\varphi_p, \qquad v_\varphi = v \sin\theta_p \sin\varphi_p. \tag{23}$$

The spherical components of the Lorentz force vector in configuration space are related to the local spherical basis of momentum space (p, θ_p, φ_p) by

$$\begin{aligned} F_p &= F_r \cos\theta_p + (F_\theta \cos\varphi_p + F_\varphi \sin\varphi_p) \sin\theta_p, \\ F_{\theta_p} &= -F_r \sin\theta_p + (F_\theta \cos\varphi_p + F_\varphi \sin\varphi_p) \cos\theta_p, \\ F_{\varphi_p} &= -F_\theta \sin\varphi_p + F_\varphi \cos\varphi_p. \end{aligned} \tag{24}$$

The substitution of the distribution function (2) and the collision integral (6) expansions and expressions (22)–(24) into the Boltzmann equation (21) yields

$$\Big\{\partial_t + v\mu\partial_r + \frac{v}{r}(1-\mu^2)^{1/2}\left[\frac{\sin\varphi_p}{\sin\theta}\partial_\varphi + \cos\varphi_p\partial_\theta + (1-\mu^2)^{1/2}\partial_\mu - \sin\varphi_p\cot\theta\,\partial_{\varphi_p}\right]$$
$$+[qE_\theta(1-\mu^2)^{1/2}\cos\varphi_p + qE_\varphi(1-\mu^2)^{1/2}\sin\varphi_p + qE_r\mu]\partial_p - [\omega_\theta\sin\varphi_p$$
$$-\omega_\varphi\cos\varphi_p + qE_\theta p^{-1}\mu\cos\varphi_p + qE_\varphi p^{-1}\mu\sin\varphi_p$$
$$-qE_r p^{-1}(1-\mu^2)^{1/2}](1-\mu^2)^{1/2}\partial_\mu + [\omega_\theta\mu\cos\varphi_p + \omega_\varphi\mu\sin\varphi_p$$
$$-(1-\mu^2)^{1/2}\omega_r - qE_\theta p^{-1}\sin\varphi_p + qE_\varphi p^{-1}\cos\varphi_p]$$
$$\times(1-\mu^2)^{-1/2}\partial_{\varphi_p}\Big\}\sum_{s=0}^{1}\sum_{n=0}^{\infty}\sum_{m=0}^{n} f_{n,m,s}P_n^m(\mu)[\delta_{s,1}\sin(m\varphi_p) + \delta_{s,0}\cos(m\varphi_p)]$$
$$= \sum_{s=0}^{1}\sum_{n=0}^{\infty}\sum_{m=0}^{n} C_n(f_{n,m,s})Y_{n,m,s}, \tag{25}$$

where the coefficients are

$$\omega_r = qvp^{-1}B_r, \qquad \omega_\theta = qvp^{-1}B_\theta, \qquad \omega_\varphi = qvp^{-1}B_\varphi.$$

Again replace expressions (12) by the linear combinations of associated Legendre functions according to formulae (A.7)–(A.11), (A.16)–(A.17) and substitute results into equation (25).

Multiplying equation (25) by $Y_{n',m',s'}(\theta_p,\varphi_p)$, integrating over all directions of momentum space $\boldsymbol{i}_p$ and using the orthogonality relationship (4) gives the following infinite hierarchy of equations for the coefficients $f_{n,m,s}$:

$$\partial_t f_{n,m,s} + \frac{1+\delta_{m,1}}{2(2n-1)}\left[\left(\frac{v}{r}\partial_\theta - (m-1)\frac{v}{r}\cot\theta + qE_\theta\partial_{p,n}^{(-)}\right)f_{n-1,m-1,s}\right.$$
$$\left.-(-1)^s\left(\frac{v}{r\sin\theta}\partial_\varphi + qE_\varphi\partial_{p,n}^{(-)}\right)f_{n-1,m-1,1-s}\right]$$
$$-\frac{(n-m-1)(n-m)}{2(2n-1)}\left[\left(\frac{v}{r}\partial_\theta + (m+1)\frac{v}{r}\cot\theta + qE_\theta\partial_{p,n}^{(-)}\right)f_{n-1,m+1,s}\right.$$
$$\left.+(-1)^s\left(\frac{v}{r\sin\theta}\partial_\varphi + qE_\varphi\partial_{p,n}^{(-)}\right)f_{n-1,m+1,1-s}\right]$$
$$+\frac{n-m}{2n-1}\left[v\partial_r - (n-1)\frac{v}{r} + qE_r\partial_{p,n}^{(-)}\right]f_{n-1,m,s}$$
$$-\frac{1+\delta_{m,1}}{2(2n+3)}\left[\left(\frac{v}{r}\partial_\theta - (m-1)\frac{v}{r}\cot\theta + qE_\theta\partial_{p,n}^{(+)}\right)f_{n+1,m-1,s}\right.$$
$$\left.-(-1)^s\left(\frac{v}{r\sin\theta}\partial_\varphi + qE_\varphi\partial_{p,n}^{(+)}\right)f_{n+1,m-1,1-s}\right]$$
$$+\frac{(n+m+1)(n+m+2)}{2(2n+3)}\left[\left(\frac{v}{r}\partial_\theta + (m+1)\frac{v}{r}\cot\theta + qE_\theta\partial_{p,n}^{(+)}\right)f_{n+1,m+1,s}\right.$$
$$\left.+(-1)^s\left(\frac{v}{r\sin\theta}\partial_\varphi + qE_\varphi\partial_{p,n}^{(+)}\right)f_{n+1,m+1,1-s}\right]$$

$$+\frac{n+m+1}{2n+3}\left[v\partial_r+(n+2)\frac{v}{r}+qE_r\partial^{(+)}_{p,n}\right]f_{n+1,m,s}$$

$$+\frac{1+\delta_{m,1}}{2}[\omega_\varphi f_{n,m-1,s}+(-1)^s\omega_\theta f_{n,m-1,1-s}]-(-1)^s m\omega_r f_{n,m,1-s}$$

$$-\frac{(n-m)(n+m+1)}{2}[\omega_\varphi f_{n,m+1,s}-(-1)^s\omega_\theta f_{n,m+1,1-s}]=C_n(f_{n,m,s}),$$

$$(n=0,1,2,\ldots,\infty;\quad m=0,1,2,\ldots,n;\quad s=0,1). \tag{26}$$

Coefficients $f_{n,m,s}$ satisfy condition (14).

5. Comparison with previous results

The non-relativistic Boltzmann equation for the distribution function in phase space is written as

$$\left[\partial_t+\boldsymbol{v}\cdot\partial_{\boldsymbol{r}}+\frac{q}{M}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\partial_{\boldsymbol{v}}\right]\tilde{f}(\boldsymbol{r},\boldsymbol{v},t)=C(\tilde{f}). \tag{27}$$

The decomposition of the distribution function in terms of spherical harmonics in velocity space with the basis of spherical coordinates $\boldsymbol{v}=(v,\theta_v,\varphi_v)$ is

$$\tilde{f}(\boldsymbol{r};v,\theta_v,\varphi_v;t)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{1}\tilde{f}_{n,m,s}(\boldsymbol{r};v;t)$$

$$\times P_n^m(\cos\theta_p)[\delta_{s,1}\sin(m\varphi_p)+\delta_{s,0}\cos(m\varphi_p)]. \tag{28}$$

The hierarchy of equations for the distribution function expansion coefficients $\tilde{f}_{n,m,s}$ for Cartesian, cylindrical and spherical geometry of configuration space results from equations (13), (20) and (26) through transformations

$$p=Mv,\qquad \partial_p=M^{-1}\partial_v,\qquad f_{n,m,s}(\boldsymbol{r};p;t)=M^3\tilde{f}_{n,m,s}(\boldsymbol{r};v;t). \tag{29}$$

Below we compare the hierarchy in cylindrical (20) and Cartesian (13) coordinates with results obtained in [5–7].

5.1. Cylindrical geometry

The hierarchy of equations for distribution function decomposition coefficients in the presence of electric field $\boldsymbol{E}=(E_r,0,E_z)$ and gradient $\partial_{\boldsymbol{r}}=(\partial_r,0,\partial_z)$ for axially symmetric geometry is given in [6]. The decomposition of the distribution function in terms of spherical harmonics in velocity space is described in this paper as follows:

$$F(r,z;v,\theta_v,\tilde{\varphi}_v-\varphi;t)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}F_{n,m}(r,z;v;t)P_n^{|m|}(\cos\theta_v)\,\mathrm{e}^{\mathrm{i}m(\tilde{\varphi}_v-\varphi)}. \tag{30}$$

As the expansion coefficients $F_{n,m}$ and $F_{n,-m}$ are complex conjugate functions

$$\mathrm{Re}(F_{n,-m})=\mathrm{Re}(F_{n,m}),\qquad \mathrm{Im}(F_{n,-m})=-\mathrm{Im}(F_{n,m}), \tag{31}$$

expression (30) can be rewritten as

$$F=\sum_{n=0}^{\infty}\mathrm{Re}(F_{n,0})P_n^0(\cos\theta_v)+2\sum_{n=1}^{\infty}\sum_{m=1}^{n}\mathrm{Re}(F_{n,m})P_n^m(\cos\theta_v)\cos[m(\tilde{\varphi}_v-\varphi)]$$

$$-2\sum_{n=1}^{\infty}\sum_{m=1}^{n}\mathrm{Im}(F_{n,m})P_n^m(\cos\theta_v)\sin[m(\tilde{\varphi}_v-\varphi)]. \tag{32}$$

It should be noted that the azimuthal angle $\tilde{\varphi}_v - \varphi$ in [6] corresponds to the azimuthal angle φ_v in the moving coordinates used in this paper. The comparison of equations (32) and (28) gives the following relations between the distribution function expansion coefficients $\tilde{f}_{n,m,s}$ introduced in our paper and $F_{n,m}$ used in [6]

$$\begin{aligned} \tilde{f}_{n,0,0} &= \mathrm{Re}(F_{n,0}) && \text{for} \quad n = 0, 1, \ldots, \infty, \\ \tilde{f}_{n,m,0} &= 2\,\mathrm{Re}(F_{n,m}) && \text{for} \quad n = 1, 2, \ldots, \infty; \quad m = 1, 2, \ldots, n, \\ \tilde{f}_{n,m,1} &= -2\,\mathrm{Im}(F_{n,m}) && \text{for} \quad n = 1, 2, \ldots, \infty; \quad m = 1, 2, \ldots, n. \end{aligned} \tag{33}$$

As the magnetic field is not taken into account in [6], then $\mathrm{Im}(F_{n,m}) = 0$, $F_{n,m} = \mathrm{Re}(F_{n,m})$ and equation (33) transforms into

$$\begin{aligned} \tilde{f}_{n,0,0} &= F_{n,0} && \text{for} \quad n = 0, 1, \ldots, \infty, \\ \tilde{f}_{n,m,0} &= 2F_{n,m} && \text{for} \quad n = 1, 2, \ldots, \infty; \quad m = 1, 2, \ldots, n. \end{aligned} \tag{34}$$

After transforming the hierarchy (20) according to formula (29), replacing the distribution function coefficients $\tilde{f}_{n,m,s}$ by $F_{n,m}$ according to formula (34), taking into account only two components of electric field $\boldsymbol{E} = (E_r, 0, E_z)$ and gradient $\partial_{\boldsymbol{r}} = (\partial_r, 0, \partial_z)$, the result obtained coincides with the hierarchy (33) in [6].

Similar transformations of equations for the distribution function expansion coefficients (B.1)–(B.16) in four-term approximation taking into account only the radial electric field $\boldsymbol{E} = (E_r, 0, 0)$ and gradient $\partial_{\boldsymbol{r}} = (\partial_r, 0, 0)$ give equations which are identical to equations (37)–(42) in [6].

The electric and magnetic fields with components $\boldsymbol{E} = (E_r, 0, 0)$, $\boldsymbol{B} = (0, 0, B_z)$ and gradient $\partial_{\boldsymbol{r}} = (\partial_r, 0, 0)$ are considered in [7]. The decomposition of the distribution function in terms of spherical harmonics in velocity space takes the form

$$\tilde{F}(r, z; v, \theta_v, \tilde{\varphi}_v - \varphi; t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{2^{|m|}} \tilde{F}_{n,m}(r, z; v; t) P_n^{|m|}(\cos\theta_v)\, \mathrm{e}^{\mathrm{i}m(\tilde{\varphi}_v - \varphi)}. \tag{35}$$

Comparing equations (35) and (30) and accounting for (33) give the following relations for expansion coefficients

$$\begin{aligned} \tilde{f}_{n,0,0} &= \mathrm{Re}(\tilde{F}_{n,0}) && \text{for} \quad n = 0, 1, \ldots, \infty, \\ \tilde{f}_{n,m,0} &= \frac{1}{2^{|m|-1}} \mathrm{Re}(\tilde{F}_{n,m}) && \text{for} \quad n = 1, 2, \ldots, \infty, \quad m = 1, 2, \ldots, n, \\ \tilde{f}_{n,m,1} &= -\frac{1}{2^{|m|-1}} \mathrm{Im}(\tilde{F}_{n,m}) && \text{for} \quad n = 1, 2, \ldots, \infty, \quad m = 1, 2, \ldots, n. \end{aligned} \tag{36}$$

After transforming the hierarchy (20) according to (29) taking into account only the field components $\boldsymbol{E} = (E_r, 0, 0)$, $\boldsymbol{B} = (0, 0, B_z)$ and gradient $\partial_{\boldsymbol{r}} = (\partial_r, 0, 0)$, introducing $\omega_z = -\Omega$ and considering $\partial_t \tilde{f}_{n,m,s} = 0$, the result obtained coincides with the hierarchy (13) in [7] if in equation (13) complex expansion coefficients are transformed to real ones. For that we take real and imaginary parts of the hierarchy (13) in [7] and replace its expansion coefficients by those introduced in our paper by formulae (36) and (31).

5.2. *Cartesian geometry*

The system consisting of 16 differential equations for distribution function expansion coefficients for the four-term approximation in the presence of electric and magnetic fields and gradients was given by Jonston [5]. A similar system of equations can be obtained from hierarchy (13) transformed with formulae (29). It can also be derived from equations (B.1)–(B.16) with formulae (29) with further transformation from cylindrical coordinates to Cartesian ones using

$$\partial_r \to \partial_x, \qquad \frac{1}{r}\partial_\varphi \to \partial_y, \qquad \frac{v}{r} \to 0. \tag{37}$$

Comparing our results with those given in [5] we see some differences in values of some coefficients and their signs. Probably there are errata in [5]. For example, in equations (13) and (14) in [5] terms $2\omega_z f_{3,2,1}$ and $2\omega_z f_{3,2,0}$ have identical signs and in equations (15) and (16) terms $3\omega_z \tilde{f}_{3,3,1}$ and $3\omega_z \tilde{f}_{3,3,0}$ also have identical signs. It follows from the hierarchy (13) that the sign must vary according to the rule

$$\partial_t \tilde{f}_{n,m,s} + \cdots - (-1)^s m\omega_z \tilde{f}_{n,m,1-s} + \cdots. \tag{38}$$

It should be noted that the same follows from hierarchy (13) in [7].

6. Application to electron transport in nitrogen at high values of E/N

Hays *et al* [9] have measured the electron-impact ionization rate coefficient in molecular nitrogen for a range of rf electric field intensity to number density ratio E/N from 450 to 12 000 Td (1 Td $= 10^{-21}$ V m^2). These measurements were made in an electrodeless cell contained in an *S*-band waveguide immersed in a dc magnetic field and subject to a pulsed rf electric field at cyclotron resonance.

They also simulated this experiment in a very simplified setup. A spatially uniform microwave field was suddenly applied to a swarm of electrons in a background gas of nitrogen molecules in the ground state and imbedded in a uniform static magnetic field aligned perpendicular to the electric field. In Cartesian components the electric and magnetic fields are given by

$$E_z = E_0 \cos(\omega_h t), \qquad B_y = 0.125\,\mathrm{T}, \tag{39}$$

where $\omega_h = qB_y\mathrm{m}^{-1}$. A spherical harmonics expansion of the non-relativistic Boltzmann equation with a two-term approximation was used and simplified to follow

$$\left\{\partial_t - \frac{1}{3v^2}\partial_v\left(v^2\frac{\mathrm{e}^2E_{\mathrm{eff}}^2}{m^2\nu_{\mathrm{tot}}(v)}\partial_v\right)\right\}\tilde{f}_0 = C(\tilde{f}_0). \tag{40}$$

The effective dc electric field E_{eff}, which includes effects of a microwave electric field and a constant magnetic field, is given by [9]

$$E_{\mathrm{eff}} = \frac{E_0}{2}\left[\frac{1}{1+(\omega_h-\omega_y)^2\nu_{\mathrm{tot}}^{-2}(v)} + \frac{1}{1+(\omega_h+\omega_y)^2\nu_{\mathrm{tot}}^{-2}(v)}\right]^{1/2}. \tag{41}$$

The total scattering frequency is

$$\nu_{\mathrm{tot}}(T) = v(T)NQ^{\mathrm{tot}}(T), \tag{42}$$

where $Q^{\mathrm{tot}}(T)$ is the total N_2 scattering cross section including elastic, inelastic and ionizing collisions [19]. For conditions of the problem $\omega_y = \omega_h = 2.199 \times 10^{10}c^{-1}$, $\nu_{\mathrm{tot}} < 2 \times 10^8 c^{-1}$ and $E_{\mathrm{eff}} \approx E_0/2$.

From relativistic formulae we obtain

$$\omega_y(T) = \omega_h\frac{mc^2}{mc^2+T}, \qquad v(T) = c\frac{\left[T\left(T+2mc^2\right)\right]^{1/2}}{T+mc^2}. \tag{43}$$

Substituting equations (42) and (43) into (41) we find ratio $2E_{\mathrm{eff}}/E_0$ as a function of electron kinetic energy *T*, which is plotted in figure 1. Relativity decreases the effective electric field E_{eff} for electrons with kinetic energy greater than 1 keV.

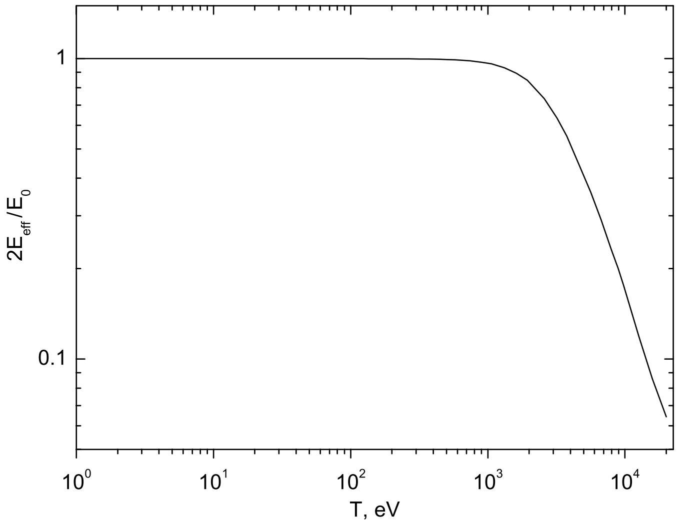

Figure 1. Ratio $2E_{\rm eff}/E_0$ calculated from equation (41) versus electron kinetic energy T.

In our calculations we used a Maxwellian initial electron distribution function with the average electron energy 0.75 eV. The pressure of N_2 was 0.018 Torr and its density was $N = 3.67 \times 10^{14}\,\text{cm}^{-3}$. Calculations were carried out for the density-normalized electric fields $E/N = 9076$ Td and $E/N = 90760$ Td. The amplitude of the rf electric field was $E_0 = 2E$.

The collision operator includes electron-neutral-molecules elastic, inelastic and ionizing collisions. In experiments described in [9] the electron density at the time when the high-frequency field was applied was $5 \times 10^6\,\text{cm}^{-3}$, and during the pulse it did not exceed $10^9\,\text{cm}^{-3}$. This permits us to ignore electron–electron and electron–ion collisions.

The infinite hierarchy of equations for the distribution function coefficients $f_{n,m,s}$ obtained in (13) takes the following form:

$$\partial_t f_{n,m,s} + \frac{n-m}{2n-1} qE_z \left(\partial_p - \frac{n-1}{p}\right) f_{n-1,m,s} + \frac{n+m+1}{2n+3} qE_z \left(\partial_p + \frac{n+2}{p}\right) f_{n+1,m,s}$$
$$+ \frac{1+\delta_{m,1}}{2}\omega_y f_{n,m-1,s} - \frac{(n-m)(n+m+1)}{2}\omega_y f_{n,m+1,s} = C_n(f_{n,m,s}),$$
$$(n = 0, 1, 2, \ldots, \infty; \quad m = 0, 1, 2, \ldots, n; \quad s = 0), \tag{44}$$

where $\omega_y = qvB_y p^{-1}$.

The collision operator coefficients $C_n\left(f_{n,m,s}(p,t)\right)$ are equal to [4, 8, 9, 17]

$$C_n(f_{n,m,s}(p,t)) = -Nv\left(Q_0^0(p) + \sum_k Q_0^k(p) + \sum_i Q_0^i(p)\right) f_{n,m,s}(p,t)$$
$$+ NvQ_n^0(p) + N\sum_k Q_n^k(p_k)v_k f_{n,m,s}(p_k,t)$$
$$+ \delta_{n,0}\frac{(p^2+m^2c^2)^{1/2}}{Mcp^{2+\xi}}\partial_p\left\{vp^{3+\xi}\left[Q_0^0(p) - Q_1^0(p)\right] f_{0,0,0}(p,t)\right\}$$

$$+\delta_{n,0} N \sum_i \int_{T+\varepsilon_i}^{2T+\varepsilon_i} \sigma_n^i(\epsilon, \epsilon - \varepsilon_i - T) v_\epsilon f_{0,0,0}(p_\epsilon, t)\, \mathrm{d}\epsilon$$

$$+\delta_{n,0} N \sum_i \int_{2T+\epsilon_i}^{T_{\max}} \sigma_n^i(\epsilon, T) v_\epsilon f_{0,0,0}(p_\epsilon, t)\, \mathrm{d}\epsilon. \qquad (45)$$

The following designations are used

$$v_k = c \frac{p_k}{\left(p_k^2 + m^2c^2\right)^{1/2}}, \qquad (46)$$

$$p_k^2 + m^2c^2 = \left[\left(p^2 + m^2c^2\right)^{1/2} + c^{-1}\varepsilon_k\right]^2, \qquad (47)$$

$$v_\epsilon = c \frac{p_\epsilon}{\left(p_\epsilon^2 + m^2c^2\right)^{1/2}}, \qquad (48)$$

$$p_\epsilon = [(mc + c^{-1}\epsilon)^2 - m^2c^2]^{1/2}, \qquad (49)$$

$$\xi = \frac{v^2}{c^2}. \qquad (50)$$

Kinetic energy T is related to momentum p by

$$(T + mc^2)^2 = p^2c^2 + m^2c^4. \qquad (51)$$

In equation (45) for decomposition coefficients of cross section Q_n^j superscripts $j = 0, k, i$ indicate types of collision process: elastic, inelastic or ionizing collisions, respectively, the same designation as in [8]; ε_k and ε_i are excitation and ionization potentials; $\sigma_0^i(\epsilon, T)$ is the differential cross section for the electron-impact ionization of N_2, (ϵ and T are kinetic energies of incident and secondary electrons, respectively); m and M are masses of electron and nitrogen molecules.

Cross sections for electron collisions with nitrogen molecules assembled in articles [8, 18–21], analytical formulae for elastic Q_n^0/Q_0^0 and inelastic Q_n^k/Q_0^k (for $n = 1, \ldots, 5$) derived in paper by Phelps *et al* [8] were used. Angular distribution of electrons produced by ionization of nitrogen molecules is supposed to be isotropic.

Truncation of the expansion (2) and the hierarchy (44)–(45) at $n = n_{\max}$ corresponds to the $(n_{\max} + 1)$-term approximation and results in $(n_{\max} + 1)(n_{\max} + 2)/2$ independent integral-differential equations and the same number of unknown $f_{n,m,s}$.

We calculated the following quantities: electron density, ionization frequency ν_i, mean electron kinetic energy $\langle T\rangle$. These quantities are expressed through the coefficient $f_{0,0,0}(p, t)$ according to

$$\langle T\rangle = \frac{4\pi}{n_e} \int T f_{0,0,0}(p, t) p^2\, \mathrm{d}p, \qquad (52)$$

$$\frac{\nu_i}{N} = \frac{4\pi}{n_e} \int v Q_0^i(p) f_{0,0,0}(p, t) p^2\, \mathrm{d}p, \qquad (53)$$

$$n_e = 4\pi \int f_{0,0,0}(p, t) p^2\, \mathrm{d}p. \qquad (54)$$

In table 1 mean electron energy $\langle T\rangle$ and density-normalized ionization frequency ν_i/N for time $Nt = 4 \times 10^7$ cm^{-3} s are given for different approximation orders. For $E/N = 9076$ Td the third-order approximation is sufficient, while for larger $E/N = 90760$ Td convergence is slower and at least the 7th order is required.

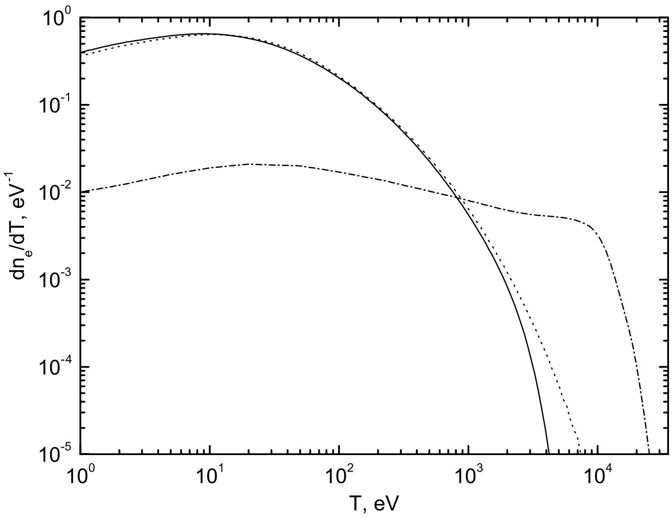

Figure 2. Calculated differential density dn_e/dT for time $Nt = 4 \times 10^7$ cm^{-3} s as a function of electron kinetic energy *T*. A normalization is chosen in which $n_e = 1$ at $t = 0$. The solid line is for $E/N = 9076$ Td and the relativistic equation. The dot line is for $E/N = 9076$ Td and the non-relativistic equation. The dash-dot line is for $E/N = 90760$ Td and the relativistic equation.

Table 1. Calculated values of parameters for different approximation orders.

E/N (Td)	n-term	$\langle T\rangle$(keV)	ν_i/N (10^{-7} cm^3 s^{-1})
9 076	2	0.193	1.237
9 076	3	0.218	1.162
9 076	4	0.215	1.161
90 760	2	5.56	1.050
90 760	3	5.76	0.930
90 760	4	5.17	1.088
90 760	5	5.42	1.013
90 760	6	5.23	1.066
90 760	7	5.31	1.043

Additional calculations using the non-relativistic Boltzmann equation with constant volume $\omega_y = \omega_h$ were carried out.

In figure 2 differential densities dn_e/dT for time $Nt = 4 \times 10^7$ cm^{-3} s as functions of electron kinetic energy T for $E/N = 9076$ Td in the four-term approximation and for $E/N = 90\,760$ Td in the seven-term approximation are plotted.

Figure 2 shows that in a dc magnetic field and pulsed rf electric field at cyclotron resonance it is important to take into account relativistic effects. Calculation results for the relativistic and non-relativistic Boltzmann equations at $E/N = 9076$ Td differ insignificantly, while at $E/N = 90760$ Td they are noticeably different. The solutions of the non-relativistic Bolzmann equation for $E/N = 90760$ Td and time $Nt = 4 \times 10^7$ cm^{-3} s show that the mean electron energy $\langle T\rangle$ and density-normalized ionization frequency ν_i/N are equal to 90 keV and 5.8×10^{-8} cm^3 s^{-1}, respectively.

The experimental value of the density-normalized ionization frequency for $E/N = 90760$ Td and time $Nt = 4 \times 10^7$ cm^{-3} s measured in paper [9] equals $8 \times 10^{-7} < \nu_i^{\text{exper.}}/N < 1.1 \times 10^{-7}$ cm^3 s^{-1}. Our calculated value equals 1.161×10^{-7} cm^3 s^{-1}. Calculations described in paper [9] gave a larger value 1.28×10^{-7} cm^3 s^{-1}.

These calculations reveal the significance of using higher order approximation of the relativistic Boltzmann equation for the described problem.

7. Conclusion

The Relativistic Boltzmann equation for the charged particles distribution function in the presence of electric and magnetic fields in weakly ionized plasma was considered. The distribution function is decomposed in terms of spherical harmonics in momentum space. A simple method of deducing an infinite hierarchy of differential equations for the distribution function expansion coefficients based on properties of associated Legendre functions was proposed. Such infinite hierarchies were obtained for Cartesian, cylindrical and spherical geometries. If we truncate them at any step we receive a finite system of equations for a finite number of distribution function expansion coefficients. By solving this system we get approximate solutions of the Boltzmann equation with any desired order. Such solutions can be used to derive transport coefficients of weakly ionized plasma, using techniques described elsewhere [4, 7–9].

We applied obtained equations to the description of electron transport in nitrogen at high values of rf electric field intensity to number density ratio E/N. The time-dependent Boltzmann equation was numerically solved for the electron cloud in weakly ionized nitrogen in the presence of a dc magnetic field of 0.125 T and a transverse rf electric field at cyclotron resonance. The influence of high-order terms approximation and relativity was studied. Results were compared with experimental data and calculations using other methods [9].

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Appendix A. Recursion relations

In [16] five recursion relations ($-1 < \mu < 1$) are given by

$$(2n+1)\mu P_n^m(\mu) - (n-m+1)P_{n+1}^m(\mu) - (n+m)P_{n-1}^m(\mu) = 0, \ (0 \leqslant m \leqslant n-1) \tag{A.1}$$

$$(\mu^2 - 1)\partial_\mu P_n^m(\mu) - (n-m+1)P_{n+1}^m(\mu) + (n+1)\mu P_n^m(\mu) = 0, \quad (0 \leqslant m \leqslant n) \tag{A.2}$$

$$P_n^{m+2}(\mu) - 2(m+1)\mu(1-\mu^2)^{-1/2}P_n^{m+1}(\mu) + [n(n+1) - m(m+1)]\,P_n^m(\mu) = 0, \quad (0 \leqslant m \leqslant n-2) \tag{A.3}$$

$$P_{n+1}^m(\mu) - P_{n-1}^m(\mu) - (2n+1)(1-\mu^2)^{1/2}P_n^{m-1}(\mu) = 0, \quad (1 \leqslant m \leqslant n-1) \tag{A.4}$$

$$(n+m)(n+m+1)P_{n-1}^m(\mu) - (n-m)(n-m+1)P_{n+1}^m(\mu) - (2n+1)(1-\mu^2)^{1/2}P_n^{m+1}(\mu) = 0, \quad (0 \leqslant m \leqslant n-1). \tag{A.5}$$

For convenience of the further notation, we assume

$$P_n^m(\mu) = 0, \quad \text{for cases} \quad m < 0 \quad \text{or} \quad n < 0 \quad \text{or} \quad n < m. \tag{A.6}$$

Taking into account expression (A.6) equations (A.1), (A.4)–(A.5) are correct for $m = n$ and equation (A.3) is correct for $m = n - 1$.

Using expressions (A.1)–(A.5) and the definition of associated Legendre functions (5), we can express terms (12) by linear combinations of associated Legendre functions.

Solving equation (A.1) with respect to $\mu P_n^m(\mu)$ and using (A.6), we can write

$$\mu P_n^m(\mu) = \frac{1}{2n+1}\left[(n-m+1)P_{n+1}^m(\mu) + (n+m)P_{n-1}^m(\mu)\right], \quad (0 \leqslant m \leqslant n). \tag{A.7}$$

Substituting equation (A.7) into equation (A.2), we find that

$$(1-\mu^2)\partial_\mu P_n^m(\mu) = \frac{1}{2n+1}\left[(n+1)(n+m)P_{n-1}^m(\mu) - n(n-m+1)P_{n+1}^m(\mu)\right], \quad (0 \leqslant m \leqslant n). \tag{A.8}$$

Solving equation (A.3) with respect to $\mu(1-\mu^2)^{-1/2}P_n^m(\mu)$ and taking into account expression (A.6), we obtain

$$\mu(1-\mu^2)^{-1/2}P_n^m(\mu) = \frac{1}{2m}\left\{P_n^{m+1}(\mu) + [n(n+1)-(m-1)m]\,P_n^{m-1}(\mu)\right\}, \quad (1 \leqslant m \leqslant n). \tag{A.9}$$

Taking into account expression (A.6), we obtain from equation (A.4) the following expression:

$$(1-\mu^2)^{1/2}P_n^m(\mu) = \frac{1}{2n+1}\left[P_{n+1}^{m+1}(\mu) - P_{n-1}^{m+1}(\mu)\right], \quad (0 \leqslant m \leqslant n). \tag{A.10}$$

Multiplying equations (A.4)–(A.5) by a factor $(1-\mu^2)^{-1/2}$, solving the new equations with respect to $(1-\mu^2)^{-1/2}P_n^m(\mu)$ and taking into account expression (A.6), we can obtain

$$(1-\mu^2)^{-1/2}P_n^m(\mu) = \frac{1}{2m}\left[P_{n+1}^{m+1}(\mu) + (n-m+1)(n-m+2)P_{n+1}^{m-1}(\mu)\right], \quad (1 \leqslant m \leqslant n). \tag{A.11}$$

Excluding expression $(1-\mu^2)^{1/2}P_n^{m-1}(\mu)$ from equations (A.4)–(A.5) and taking into account expression (A.6), we can obtain

$$P_n^m(\mu) = \left[P_{n-2}^m(\mu) + (n+m-3)(n+m-2)P_{n-2}^{m-2}(\mu) - (n-m+1)(n-m+2)P_n^{m-2}(\mu)\right], \quad (2 \leqslant m \leqslant n). \tag{A.12}$$

Multiplying equation (A.8) by a factor $(1-\mu^2)^{-1/2}$ and substituting expression $(1-\mu^2)^{-1/2}P_n^m(\mu)$ from equation (A.11) into the received equation, we can obtain

$$(1-\mu^2)^{1/2}\partial_\mu P_n^m(\mu) = -\frac{n(n-m+1)}{2m(2n+1)}\left[P_{n+2}^{m+1}(\mu) + (n-m+2)(n-m+3)P_{n+2}^{m-1}(\mu)\right] + \frac{(n+1)(n+m)}{2m(2n+1)}\left[P_n^{m+1}(\mu) + (n-m)(n-m+1)P_n^{m-1}(\mu)\right], \quad (1 \leqslant m \leqslant n). \tag{A.13}$$

Transforming expression (A.13) according to equation (A.12), we obtain

$$(1-\mu^2)^{1/2}\partial_\mu P_n^m(\mu) = \tfrac{1}{2}\left[P_n^{m+1}(\mu) - (n+m)(n-m+1)P_n^{m-1}(\mu)\right], \quad (1 \leqslant m \leqslant n). \tag{A.14}$$

From the definition of associated Legendre functions (5) it follows

$$(1-\mu^2)^{1/2}\partial_\mu P_n^0(\mu) = P_n^1(\mu), \quad (n \geqslant 0). \tag{A.15}$$

Equations (A.14) and (A.15) can be united as

$$(1-\mu^2)^{1/2}\partial_\mu P_n^m(\mu) = \frac{1+\delta_{m,0}}{2} P_n^{m+1}(\mu) - \frac{(n+m)(n-m+1)}{2} P_n^{m-1}(\mu),$$
$$(0 \leqslant m \leqslant n). \tag{A.16}$$

Multiplying equation (A.16) by a factor μ, substituting into the received equation expression $\mu P_n^m(\mu)$ from equation (A.7), we find that

$$\mu(1-\mu^2)^{1/2}\partial_\mu P_n^m(\mu) = \frac{1+\delta_{m,0}}{2(2n+1)}\left[(n-m)P_{n+1}^{m+1}(\mu) + (n+m+1)P_{n-1}^{m+1}(\mu)\right]$$
$$- \frac{(n+m)(n-m+1)}{2(2n+1)}\left[(n-m+2)P_{n+1}^{m-1}(\mu) + (n+m-1)P_{n-1}^{m-1}(\mu)\right]$$
$$(0 \leqslant m \leqslant n). \tag{A.17}$$

Appendix B. Four-term approximation

Truncation of the hierarchy (20) at $n = 3$ in the cylindrical basis of configuration space results in 16 independent equations given below,

$$\partial_t f_{0,0,0} + \frac{1}{3}\left[v\partial_z + qE_z\left(\partial_p + \frac{2}{p}\right)\right] f_{1,0,0} + \frac{1}{3}\left[v\left(\partial_r + \frac{1}{r}\right) + qE_r\left(\partial_p + \frac{2}{p}\right)\right] f_{1,1,0}$$
$$+ \frac{1}{3}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{2}{p}\right)\right] f_{1,1,1} = C_{0,0,0}, \tag{B.1}$$

$$\partial_t f_{1,0,0} + \left(v\partial_z + qE_z\partial_p\right) f_{0,0,0} + \frac{2}{5}\left[v\partial_z + qE_z\left(\partial_p + \frac{3}{p}\right)\right] f_{2,0,0}$$
$$+ \frac{3}{5}\left[v\left(\partial_r + \frac{1}{r}\right) + qE_r\left(\partial_p + \frac{3}{p}\right)\right] f_{2,1,0} + \frac{3}{5}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{3}{p}\right)\right] f_{2,1,1}$$
$$- \omega_\varphi f_{1,1,0} + \omega_r f_{1,1,1} = C_{1,0,0}, \tag{B.2}$$

$$\partial_t f_{1,1,0} + \left(v\partial_r + qE_r\partial_p\right) f_{0,0,0} - \frac{1}{5}\left[v\partial_r + qE_r\left(\partial_p + \frac{3}{p}\right)\right] f_{2,0,0}$$
$$+ \frac{3}{5}\left[v\partial_z + qE_z\left(\partial_p + \frac{3}{p}\right)\right] f_{2,1,0} + \frac{6}{5}\left[v\left(\partial_r + \frac{2}{r}\right) + qE_r\left(\partial_p + \frac{3}{p}\right)\right] f_{2,2,0}$$
$$+ \frac{6}{5}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{3}{p}\right)\right] f_{2,2,1} + \omega_\varphi f_{1,0,0} - \omega_z f_{1,1,1} = C_{1,1,0}, \tag{B.3}$$

$$\partial_t f_{1,1,1} + \left(\frac{v}{r}\partial_\varphi + qE_\varphi\partial_p\right) f_{0,0,0} - \frac{1}{5}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{3}{p}\right)\right] f_{2,0,0}$$
$$+ \frac{3}{5}\left[v\partial_z + qE_z\left(\partial_p + \frac{3}{p}\right)\right] f_{2,1,1} - \frac{6}{5}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{3}{p}\right)\right] f_{2,2,0} + \frac{6}{5}$$
$$\times\left[v\left(\partial_r + \frac{2}{r}\right) + qE_r\left(\partial_p + \frac{3}{p}\right)\right] f_{2,2,1} - \omega_r f_{1,0,0} + \omega_z f_{1,1,0} = C_{1,1,1}, \tag{B.4}$$

$$\partial_t f_{2,0,0} + \frac{2}{3}\left[v\partial_z + qE_z\left(\partial_p - \frac{1}{p}\right)\right] f_{1,0,0} - \frac{1}{3}\left[v\left(\partial_r + \frac{1}{r}\right) + qE_r\left(\partial_p - \frac{1}{p}\right)\right] f_{1,1,0}$$
$$- \frac{1}{3}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p - \frac{1}{p}\right)\right] f_{1,1,1}, + \frac{3}{7}\left[v\partial_z + qE_z\left(\partial_p + \frac{4}{p}\right)\right] f_{3,0,0}$$
$$+ \frac{6}{7}\left[v\left(\partial_r + \frac{1}{r}\right) + qE_r\left(\partial_p + \frac{4}{p}\right)\right] f_{3,1,0} + \frac{6}{7}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{4}{p}\right)\right] f_{3,1,1}$$
$$- 3\omega_\varphi f_{2,1,0} + 3\omega_r f_{2,1,1} = C_{2,0,0}, \tag{B.5}$$

$$\partial_t f_{2,1,0} + \frac{1}{3}\left[v\partial_r + qE_r\left(\partial_p - \frac{1}{p}\right)\right] f_{1,0,0} + \frac{1}{3}\left[v\partial_z + qE_z\left(\partial_p - \frac{1}{p}\right)\right] f_{1,1,0}$$
$$- \frac{1}{7}\left[v\partial_r + qE_r\left(\partial_p + \frac{4}{p}\right)\right] f_{3,0,0}, + \frac{4}{7}\left[v\partial_z + qE_z\left(\partial_p + \frac{4}{p}\right)\right] f_{3,1,0}$$
$$+ \frac{10}{7}\left[v\left(\partial_r + \frac{2}{r}\right) + qE_r\left(\partial_p + \frac{4}{p}\right)\right] f_{3,2,0} + \frac{10}{7}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{4}{p}\right)\right] f_{3,2,1}$$
$$+ \omega_\varphi f_{2,0,0} - \omega_z f_{2,1,1} - 2\omega_\varphi f_{2,2,0} + 2\omega_r f_{2,2,1} = C_{2,1,0}, \tag{B.6}$$

$$\partial_t f_{2,1,1} + \frac{1}{3}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p - \frac{1}{p}\right)\right] f_{1,0,0} + \frac{1}{3}\left[v\partial_z + qE_z\left(\partial_p - \frac{1}{p}\right)\right] f_{1,1,1}$$
$$- \frac{1}{7}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{4}{p}\right)\right] f_{3,0,0} + \frac{4}{7}\left[v\partial_z + qE_z\left(\partial_p + \frac{4}{p}\right)\right] f_{3,1,1}$$
$$- \frac{10}{7}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{4}{p}\right)\right] f_{3,2,0} + \frac{10}{7}\left[v\left(\partial_r + \frac{2}{r}\right) + qE_r\left(\partial_p + \frac{4}{p}\right)\right] f_{3,2,1}$$
$$- \omega_r f_{2,0,0} + \omega_z f_{2,1,0} - 2\omega_r f_{2,2,0} - 2\omega_\varphi f_{2,2,1} = C_{2,1,1}, \tag{B.7}$$

$$\partial_t f_{2,2,0} + \frac{1}{6}\left[v\left(\partial_r - \frac{1}{r}\right) + qE_r\left(\partial_p - \frac{1}{p}\right)\right] f_{1,1,0} - \frac{1}{6}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p - \frac{1}{p}\right)\right] f_{1,1,1}$$
$$- \frac{1}{14}\left[v\left(\partial_r - \frac{1}{r}\right) + qE_r\left(\partial_p + \frac{4}{p}\right)\right] f_{3,1,0} + \frac{1}{14}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{4}{p}\right)\right] f_{3,1,1}$$
$$+ \frac{5}{7}\left[v\partial_z + qE_z\left(\partial_p + \frac{4}{p}\right)\right] f_{3,2,0} + \frac{15}{7}\left[v\left(\partial_r + \frac{3}{r}\right) + qE_r\left(\partial_p + \frac{4}{p}\right)\right] f_{3,3,0}$$
$$+ \frac{15}{7}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{4}{p}\right)\right] f_{3,3,1} + \frac{1}{2}\omega_\varphi f_{2,1,0} + \frac{1}{2}\omega_r f_{2,1,1}$$
$$- 2\omega_z f_{2,2,1} = C_{2,2,0}, \tag{B.8}$$

$$\partial_t f_{2,2,1} + \frac{1}{6}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p - \frac{1}{p}\right)\right] f_{1,1,0} + \frac{1}{6}\left[v\left(\partial_r - \frac{1}{r}\right) + qE_r\left(\partial_p - \frac{1}{p}\right)\right] f_{1,1,1}$$
$$- \frac{1}{14}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{4}{p}\right)\right] f_{3,1,0} - \frac{1}{14}\left[v\left(\partial_r - \frac{1}{r}\right) + qE_r\left(\partial_p + \frac{4}{p}\right)\right] f_{3,1,1}$$
$$+ \frac{5}{7}\left[v\partial_z + qE_z\left(\partial_p + \frac{4}{p}\right)\right] f_{3,2,1} - \frac{15}{7}\left[\frac{v}{r}\partial_\varphi + qE_\varphi\left(\partial_p + \frac{4}{p}\right)\right] f_{3,3,0}$$

$$+\frac{15}{7}\left[v\left(\partial_r+\frac{3}{r}\right)+qE_r\left(\partial_p+\frac{4}{p}\right)\right]f_{3,3,1}-\frac{1}{2}\omega_r f_{2,1,0}+\frac{1}{2}\omega_\varphi f_{2,1,1}$$
$$+2\omega_z f_{2,2,0}=C_{2,2,1}, \tag{B.9}$$

$$\partial_t f_{3,0,0}+\frac{3}{5}\left[v\partial_z+qE_z\left(\partial_p-\frac{2}{p}\right)\right]f_{2,0,0}-\frac{3}{5}\left[v\left(\partial_r+\frac{1}{r}\right)+qE_r\left(\partial_p-\frac{2}{p}\right)\right]f_{2,1,0}$$
$$-\frac{3}{5}\left[\frac{v}{r}\partial_\varphi+qE_\varphi\left(\partial_p-\frac{2}{p}\right)\right]f_{2,1,1}-6\omega_\varphi f_{3,1,0}+6\omega_r f_{3,1,1}=C_{3,0,0}, \tag{B.10}$$

$$\partial_t f_{3,1,0}+\frac{1}{5}\left[v\partial_r+qE_r\left(\partial_p-\frac{2}{p}\right)\right]f_{2,0,0}+\frac{2}{5}\left[v\partial_z+qE_z\left(\partial_p-\frac{2}{p}\right)\right]f_{2,1,0}$$
$$-\frac{1}{5}\left[v\left(\partial_r+\frac{2}{r}\right)+qE_r\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,0}-\frac{1}{5}\left[\frac{v}{r}\partial_\varphi+qE_\varphi\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,1}$$
$$+\omega_\varphi f_{3,0,0}-\omega_z f_{3,1,1}-5\omega_\varphi f_{3,2,0}+5\omega_r f_{3,2,1}=C_{3,1,0}, \tag{B.11}$$

$$\partial_t f_{3,1,1}+\frac{1}{5}\left[\frac{v}{r}\partial_\varphi+qE_\varphi\left(\partial_p-\frac{2}{p}\right)\right]f_{2,0,0}+\frac{2}{5}\left[v\partial_z+qE_z\left(\partial_p-\frac{2}{p}\right)\right]f_{2,1,1}$$
$$+\frac{1}{5}\left[\frac{v}{r}\partial_\varphi+qE_\varphi\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,0}-\frac{1}{5}\left[v\left(\partial_r+\frac{2}{r}\right)+qE_r\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,1}$$
$$-\omega_r f_{3,0,0}+\omega_z f_{3,1,0}-5\omega_r f_{3,2,0}-5\omega_\varphi f_{3,2,1}=C_{3,1,1}, \tag{B.12}$$

$$\partial_t f_{3,2,0}+\frac{1}{10}\left[v\left(\partial_r-\frac{1}{r}\right)+qE_r\left(\partial_p-\frac{2}{p}\right)\right]f_{2,1,0}-\frac{1}{10}\left[\frac{v}{r}\partial_\varphi+qE_\varphi\left(\partial_p-\frac{2}{p}\right)\right]f_{2,1,1}$$
$$+\frac{1}{5}\left[v\partial_z+qE_z\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,0}+\frac{1}{2}\omega_\varphi f_{3,1,0}+\frac{1}{2}\omega_r f_{3,1,1}-2\omega_z f_{3,2,1}$$
$$-3\omega_\varphi f_{3,3,0}+3\omega_r f_{3,3,1}=C_{3,2,0}, \tag{B.13}$$

$$\partial_t f_{3,2,1}+\frac{1}{10}\left[\frac{v}{r}\partial_\varphi+qE_\varphi\left(\partial_p-\frac{2}{p}\right)\right]f_{2,1,0}+\frac{1}{10}\left[v\left(\partial_r-\frac{1}{r}\right)+qE_r\left(\partial_p-\frac{2}{p}\right)\right]f_{2,1,1}$$
$$+\frac{1}{5}\left[v\partial_z+qE_z\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,1}-\frac{1}{2}\omega_r f_{3,1,0}+\frac{1}{2}\omega_\varphi f_{3,1,1}+2\omega_z f_{3,2,0}$$
$$-3\omega_r f_{3,3,0}-3\omega_\varphi f_{3,3,1}=C_{3,2,1}, \tag{B.14}$$

$$\partial_t f_{3,3,0}+\frac{1}{10}\left[v\left(\partial_r-\frac{2}{r}\right)+qE_r\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,0}-\frac{1}{10}\left[\frac{v}{r}\partial_\varphi+qE_\varphi\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,1}$$
$$+\frac{1}{2}\omega_\varphi f_{3,2,0}+\frac{1}{2}\omega_r f_{3,2,1}-3\omega_z f_{3,3,1}=C_{3,3,0}, \tag{B.15}$$

$$\partial_t f_{3,3,1}+\frac{1}{10}\left[\frac{v}{r}\partial_\varphi+qE_\varphi\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,0}+\frac{1}{10}\left[v\left(\partial_r-\frac{2}{r}\right)+qE_r\left(\partial_p-\frac{2}{p}\right)\right]f_{2,2,1}$$
$$-\frac{1}{2}\omega_r f_{3,2,0}+\frac{1}{2}\omega_\varphi f_{3,2,1}+3\omega_z f_{3,3,0}=C_{3,3,1}. \tag{B.16}$$

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